Visualising actions, computing cost, and fixed price for $G \times \mathbb{Z}$

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Executive summary

The study of essentially free pmp actions $G \curvearrowright (X, \mu)$ of lcsc groups is equivalent to the study of *invariant point processes on G*.

This is due to a fundamental relationship between *weakly* lacunary sections and point processes. I will explain this equivalence.

In particular, we can exploit this relationship to give the first new technique to compute cost in the nondiscrete case, for $G \times \mathbb{Z}$ and other groups.

Setup

Throughout, G will be a locally compact second countable (lcsc) group.

We will mostly assume G is nondiscrete, noncompact, and unimodular.

Once a Haar measure λ on G is fixed, it is possible to define the *cost* of essentially free pmp actions $G \curvearrowright (X, \mu)$ on standard Borel spaces by using lacunary sections.

Lacunary section review

Let $G \curvearrowright X$ be a Borel action on a standard Borel space.

A *lacunary section* is a Borel subset $Y \subset X$ which meets every orbit, and such that there exists an identity neighbourhood $U \subseteq G$ such that $Uy \cap Y = \{y\}$ for all $y \in Y$.

Fact: lacunary sections always exist. *(See Kechris for most general statement + history)*

This implies $Gy \cap Y$ is *countable*, so the orbit equivalence relation of $G \curvearrowright X$ restricts to a countable Borel equivalence relation $\mathscr{R}_Y$ (cber) on Y.

If there is a G-invariant probability measure μ on X for which the action $G \curvearrowright (X, \mu)$ is *essentially free*, then there is a probability measure μ_Y on Y such that $(Y, \mathscr{R}_Y, \mu_Y)$ is a quasi-pmp cber (and pmp if G is *unimodular*). Sometimes called a *cross-section equivalence relation*.

You can define $\mathrm{cost}(G \curvearrowright (X, \mu))$ to be $\mathrm{cost}(Y, \mathscr{R}_Y, \mu_Y)$ *(normalised by its intensity)*.

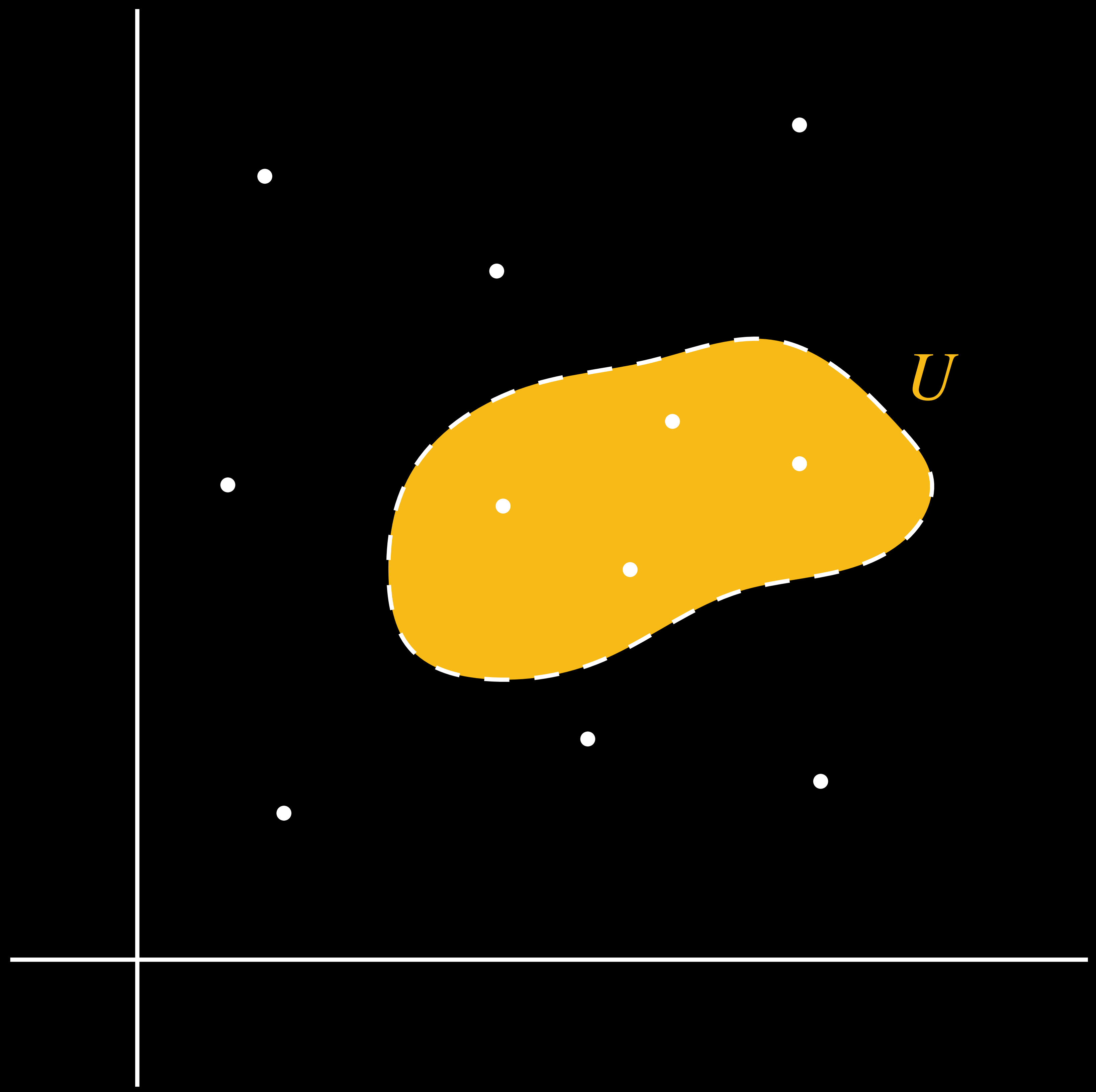
The *configuration space* action $G \curvearrowright \mathbb{M}$

Let $\mathbb{M} = \{\omega \subset G \mid \omega \text{ is locally finite}\} \subset 2^G$, where "locally finite" is with reference to a left-invariant *proper* metric on G

There is a natural shift action $G \curvearrowright \mathbb{M}$

We equip $\mathbb{M}$ with the smallest σ-algebra that makes the following *point counting* functions measurable: for $U \subseteq G$ Borel, define

$$\begin{cases} N_U : \mathbb{M} \to \mathbb{N}_0 \cup \{\infty\} \\ N_U(\omega) = |\omega \cap U| \end{cases}$$



Proper means closed balls are compact

A lacunary section for $G \curvearrowright \mathbb{M}$?

Recall: $\mathbb{M} = \{\omega \subset G \mid \omega \text{ is locally finite}\}$

The *rooted configuration space* is $\mathbb{M}_0 = \{\omega \in \mathbb{M} \mid 0 \in \omega\}$, where $0 \in G$ denotes the identity element.

Observe that if $g \in \omega$, then $0 \in g^{-1}\omega$, and so $g^{-1}\omega \in \mathbb{M}_0$. So $G\mathbb{M}_0 = \mathbb{M} \backslash \{\emptyset\}$, and $\mathbb{M}_0$ meets *essentially every* orbit of $G \curvearrowright \mathbb{M}$.

This calculation also shows that $U\omega \cap \mathbb{M}_0 = \{u^{-1}\omega \mid u \in \omega \cap U\}$.

In particular, if U is any *bounded* identity neighbourhood, then $\left|U\omega \cap \mathbb{M}_0\right|$ is always *finite* for any $\omega \in \mathbb{M}_0$.

Thus $\mathbb{M}_0$ is a *"weakly lacunary"* section for $G \curvearrowright \mathbb{M}$. ***This is the only kind of example that exists.***

For "point process actions", *weakly lacunary* is the natural notion, **not** lacunary.

Point process actions

A *point process on G* is a probability measure μ on $\mathbb{M}$.

That is, a *random discrete subset* of G.

It is *invariant* if it is an invariant measure for the action $G \curvearrowright \mathbb{M}$.

Speaking more properly, a point process is a *random element* $\Pi \in \mathbb{M}$. I will try avoid this random element terminology.

Point process examples

- *Lattice shifts*. If $\Gamma < G$ is a lattice, then view G/Γ as a subset of $\mathbb{M}$

- The *Poisson point process*. It is the nondiscrete analogue of *Bernoulli percolation* $\mathtt{Ber}(p) = (p\delta_1 + q\delta_0)^{\otimes\Gamma}$ on $\{0,1\}^\Gamma$

- Every lacunary section $Y \subset X$ of a free pmp action $G \curvearrowright (X, \mu)$ naturally induces an "orbit viewing factor map" $V : X \to \mathbb{M}$ *(next slide)*, and the push forward measure $V_*\mu$ is an invariant point process

Fact: Every free pmp action is isomorphic to a point process. *(More generally, nonfree actions are some kind of "bundle of point processes")*

Visualising lacunary sections

Fix a lacunary section $Y \subset X$ for a *free* action $G \curvearrowright X$.

We identify each orbit Gx with G via $g \mapsto gx$.

We study orbit intersections $Gx \cap Y$ under this identification. Formally, define the *orbit viewing map* $V : X \to 2^G$ by

$V(x) := \{g \in G \mid g^{-1}x \in Y\}$ *(the inverse makes it equivariant)*

By the lacunary property, V takes values in $\mathbb{M} = \{\omega \subset G \mid \omega \text{ is locally finite}\}$.

Observe that $Y = V^{-1}(\mathbb{M}_0)$: since $0 \in V(x)$ *iff* $x \in Y$.

In short: a choice of lacunary section induces a factor map $X \to \mathbb{M}$, and we recover the lacunary section as the pre-image of the *canonical section* $\mathbb{M}_0$.

Current state

- A choice of lacunary section $Y \subset X$ for a free action $G \curvearrowright (X, \mu)$ induces a factor map $V : X \to \mathbb{M}$ and hence an invariant point process $V_* \mu$

- With some work, you can construct a Borel *injection* $\mathscr{V} : X \to \mathbb{M}$, and hence $G \curvearrowright (X, \mu)$ and $G \curvearrowright (\mathbb{M}, \mathscr{V}_* \mu)$ are *isomorphic actions*

For this reason, I will now speak just of point process actions.

The *Palm equivalence relation*

If μ is an invariant point process *"of finite intensity"*, then there is a natural measure μ_0 on the weakly lacunary section $\mathbb{M}_0$ such that $(\mathbb{M}_0, \mathscr{R}|_{\mathbb{M}_0}, \mu_0)$ is a quasi-pmp cber (and pmp if G is unimodular), where $\mathscr{R}$ is the orbit equivalence relation of $G \curvearrowright \mathbb{M}$.

The measure μ_0 comes from probability theory and is known as the *Palm measure* and we call $(\mathbb{M}_0, \mathscr{R}|_{\mathbb{M}_0}, \mu_0)$ the *Palm equivalence relation* of μ.

It admits a reasonably simple algebraic definition, but I will use Palm-free definitions for interests of time.

If you are familiar with *cross-section equivalence relations*, it's the same thing *(more elegant imo)*

To define *cost* I must explain the *intensity* of a point process, what a *graphing* of a point process is, and how to measure the *size* of a graphing.

Intensity of an invariant point process $G \curvearrowright (\mathbb{M}, \mu)$

Recall that for $U \subseteq G$ the function $\begin{cases} N_U : \mathbb{M} \to \mathbb{N}_0 \cup \{\infty\} \\ N_U(\omega) = |\omega \cap U| \end{cases}$ is measurable.

Fix $U \subseteq G$ with $0 < \lambda(U) < \infty$, and define intensity$(\mu) := \dfrac{1}{\lambda(U)} \displaystyle\int_{\mathbb{M}} \left|\omega \cap U\right| d\mu(\omega) = \dfrac{\mathbb{E}_\mu\left[N_U\right]}{\lambda(U)}$

Because μ is *invariant*, the function $U \mapsto \displaystyle\int_{\mathbb{M}} \left|\omega \cap U\right| d\mu(\omega)$ defines a *right invariant* Haar measure on G, which shows that the intensity is well-defined. It scales linearly with the choice of Haar measure.

Examples

- A process has zero intensity *if and only if* it is trivial (that is, $\mu = \delta_{\varnothing}$)
- The intensity of a lattice shift $G \curvearrowright G/\Gamma$ is $1/\text{covol}(\Gamma)$

Factor graphs: the analogue of graphings

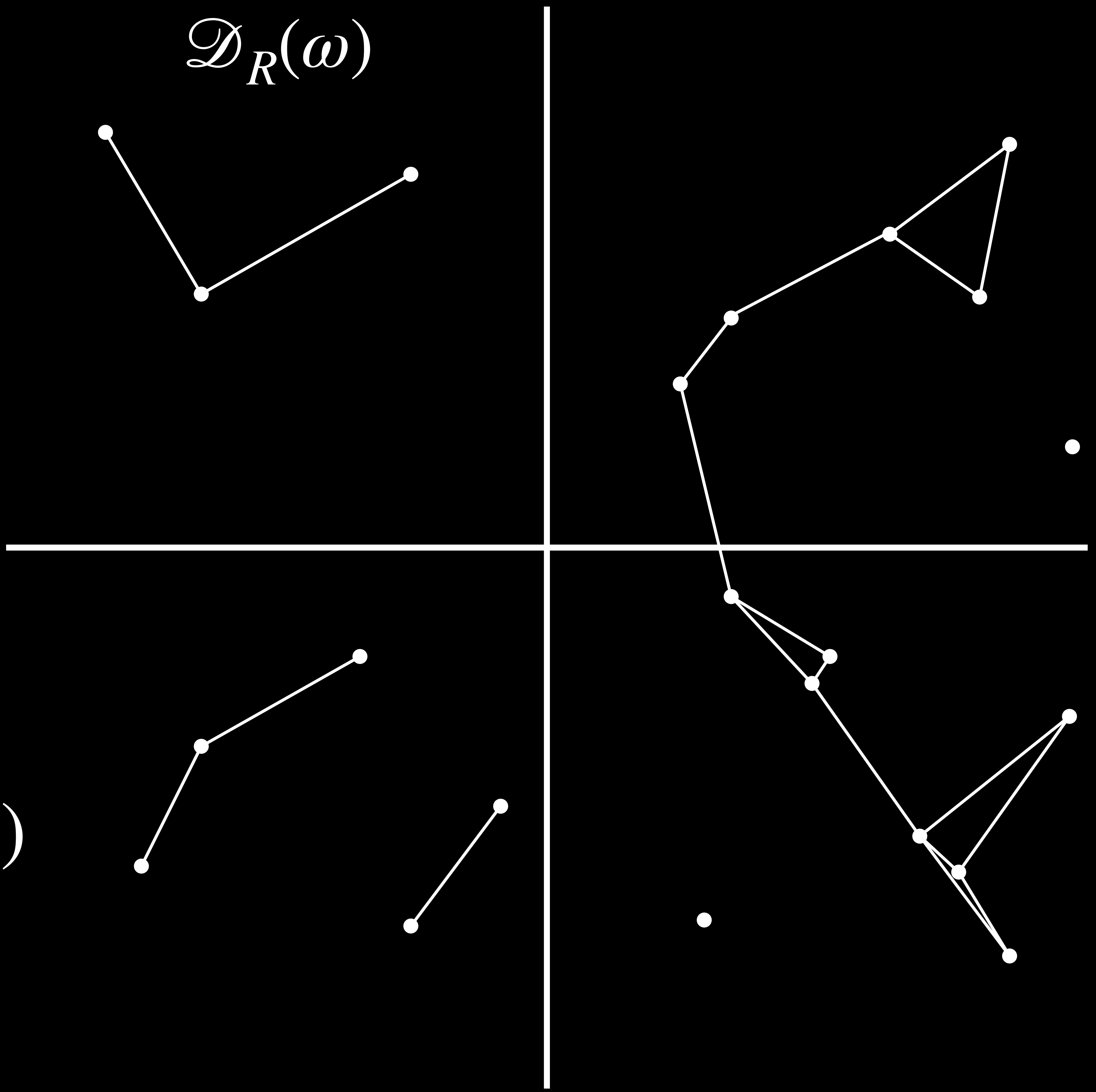
The *distance-R* factor graph on $\omega \in \mathbb{M}$ has vertex set ω and edge set

$$\mathscr{D}_R(\omega) = \{(g, h) \in \omega \times \omega \mid d(g, h) < R\}$$

Note that it is *deterministic* given the input

It's *equivariantly* defined: $\mathscr{D}_R(g\omega) = g\mathscr{D}_R(\omega)$

(we use a metric which is left-invariant)



Factor graphs, formally

Let Graph(G) denote the space of *abstractly embedded graphs in G*. Formally,

$$\mathrm{Graph}(G) = \{(V, E) \in \mathbb{M}(G) \times \mathbb{M}(G \times G) \mid E \subseteq V \times V \text{ and } E = E^{-1}\}$$

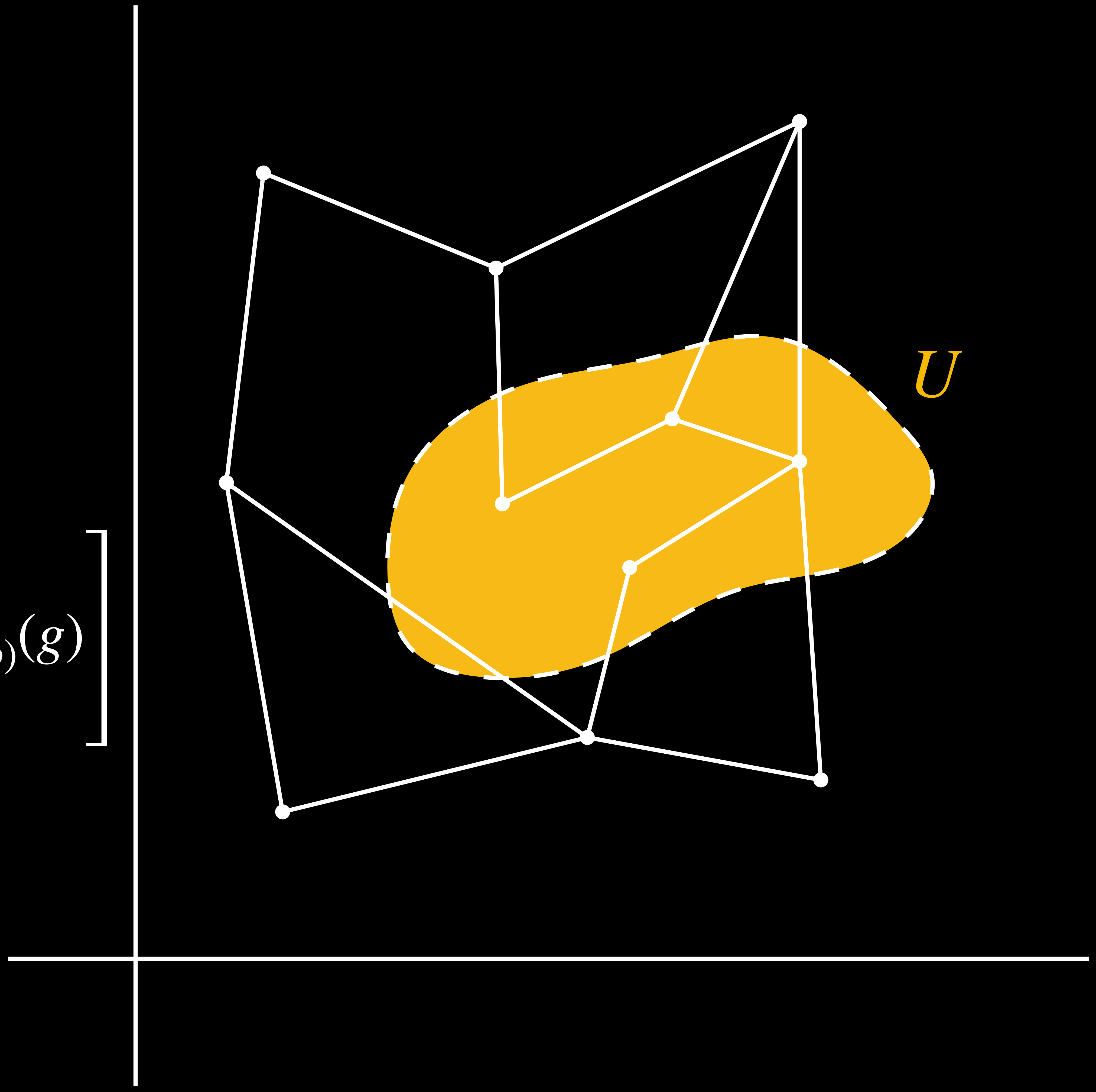
It is a G-space in its own right. A (Borel) *factor graph* is a measurable and equivariant map $\mathscr{G} : \mathbb{M}(G) \to \mathrm{Graph}(G)$ such that the vertex set of $\mathscr{G}(\omega)$ is ω.

The *average degree* of a factor graph

Let μ be an invariant point process and $\mathscr{G}$ a factor graph. The *average degree* of $\mathscr{G}$ (with respect to μ) is

$$\overleftrightarrow{\mu}_0(\mathscr{G}) = \frac{1}{2}\int_{\mathbb{M}} \sum_{g \in U \cap \omega} \deg_{\mathscr{G}(\omega)}(g) d\mu(\omega) = \frac{1}{2}\mathbb{E}\left[\sum_{g \in U \cap \omega} \deg_{\mathscr{G}(\omega)}(g)\right]$$

where $U \subseteq G$ is any set of unit volume.



Cost

The *cost* of an invariant point process μ is defined by

$$\operatorname{cost}(\mu) - 1 = \inf_{\mathscr{G}} \left[\overleftrightarrow{\mu}_0(\mathscr{G}) - \operatorname{intensity}(\mu) \right]$$

where the infimum ranges over all *connected* factor graphs $\mathscr{G}$

It's not immediately apparent from the above formula, but $(\operatorname{cost} - 1)$ scales with the choice of Haar measure λ

For *lattice shifts*, $\operatorname{cost}(G \curvearrowright G/\Gamma) - 1 = \dfrac{d(\Gamma) - 1}{\operatorname{covol}(\Gamma)}$, where $d(\Gamma)$ denotes the *rank* of Γ (minimum size of a generating set)

Cost vs. factors

If $\Phi : \mathbb{M} \to \mathbb{M}$ is a factor map and μ is a point process, then $\text{cost}(\mu) \le \text{cost}(\Phi_* \mu)$

In particular, the cost is an *isomorphism invariant* (can show it's an *OE invariant* in an appropriate sense)

This follows from Gaboriau's induction formula for cost (you can also prove it directly)

Factor of IID cost

I stress that factor graphs should be *deterministically defined*.

A (perhaps less well-known?) fact is that, as far as computing the cost is concerned, you're allowed to add *factor of IID* randomness.

Discrete example [Tucker-Drob]
If $\Gamma \curvearrowright^{\alpha} (X, \mu)$ is a *free* pmp action and $\Gamma \curvearrowright^{\beta} [0,1]^{\Gamma}$ is the *Bernoulli shift*, then α is weakly equivalent to $\alpha \times \beta$. In particular, $\text{cost}(X, \mu) = \text{cost}(X \times [0,1]^{\Gamma})$. That is, the cost of a free action is equal to the cost of its *Bernoulli extension*.

This should be true for any pmp cber (Abért-M. prove it for point processes).

If μ is any *free* point process, then it has the same cost as its *Bernoulli extension* $[0,1]^{\mu}$

Corollary: the Poisson point process has *maximal cost* (since it's a factor of *any* Bernoulli extension)

Bernoulli extensions

If $(X, \mathscr{R}, \mu)$ is a pmp cber, then it admits a *class bijective* *Bernoulli extension* $(\widetilde{X}, \widetilde{\mathscr{R}}, \widetilde{\mu}) \to (X, \mathscr{R}, \mu)$

Formally:

- $\widetilde{X}$ consists of pairs (x, f) where $x \in X$ and $f : [x]_{\mathscr{R}} \to [0,1]$ is a labelling of its equivalence class
- We declare $(x, f)\widetilde{\mathscr{R}}(x', f')$ if $x\mathscr{R}x'$ and $f = f'$
- For $A \subseteq \widetilde{X}$, we set $\widetilde{\mu}(A) = \int_X \eta_x(A) d\mu(x)$, where $\eta_x = \mathtt{Leb}^{[x]_{\mathscr{R}}}$

Informally
It's X but every point $x \in X$ has IID $\mathtt{Unif}[0,1]$ labels on the points of its equivalence class

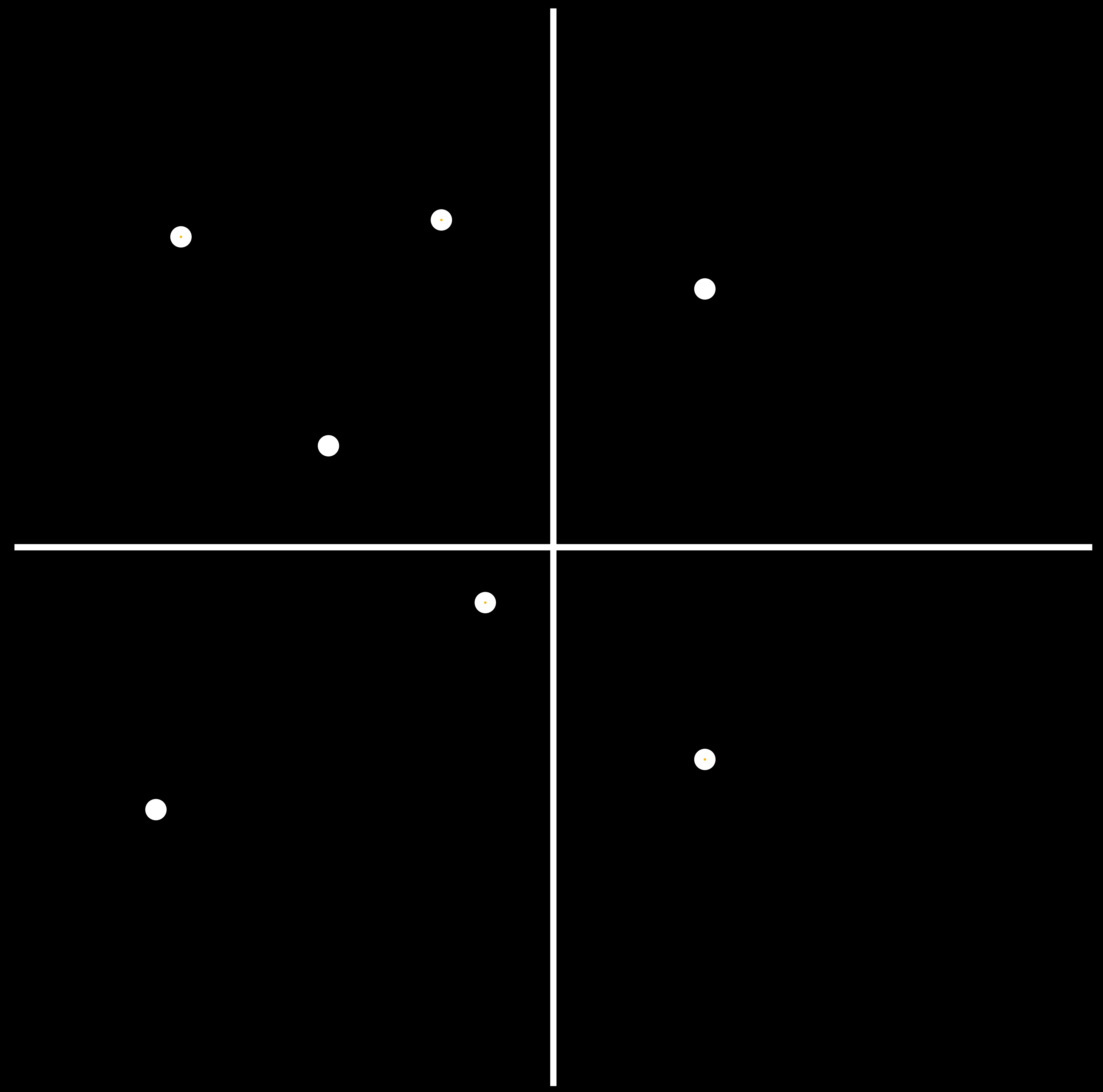
IID markings of point processes

In general, every *class bijective extension* $(\widetilde{Y}, \widetilde{\mathscr{R}}, \widetilde{\mu_Y}) \to (Y, \mathscr{R}, \mu_Y)$ of a lacunary section arises as a lacunary section of an extension $G \curvearrowright (\widetilde{X}, \mu) \to G \curvearrowright (X, \mu)$

In particular, every point process μ has a *Bernoulli extension* or *IID marking*, which I will denote by $[0,1]^\mu$.

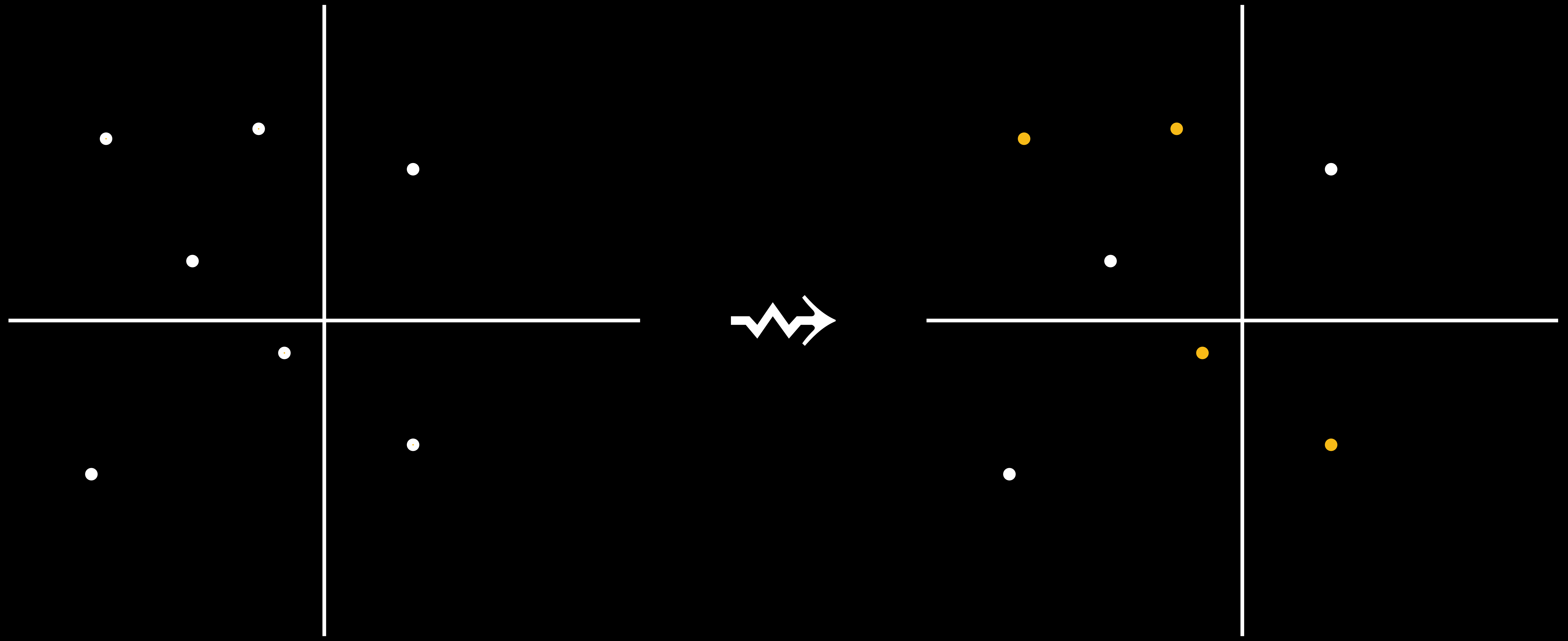
Informal example: {H, T}-IID of a point process μ

First sample the process…



Informal example: {H, T}-IID of a point process μ

First sample the process…



…*then* each point independently tosses a coin heads or tails

The Ξ-*configuration space* action $G \curvearrowright \Xi^{\mathbb{M}}$

Let Ξ denote a complete separable metric *space of marks* (think of $\{\texttt{Heads}, \texttt{Tails}\}$ or $[0,1]$).

A Ξ-*marked configuration* is a discrete subset ω of G where every point is labelled by an element of Ξ. Formally, let

$$\Xi^{\mathbb{M}} = \{\omega \in \mathbb{M}(G \times \Xi) \mid \text{ if } (g, \xi) \in \omega \text{ and } (g, \xi') \in \omega \text{ then } \xi = \xi'\}$$

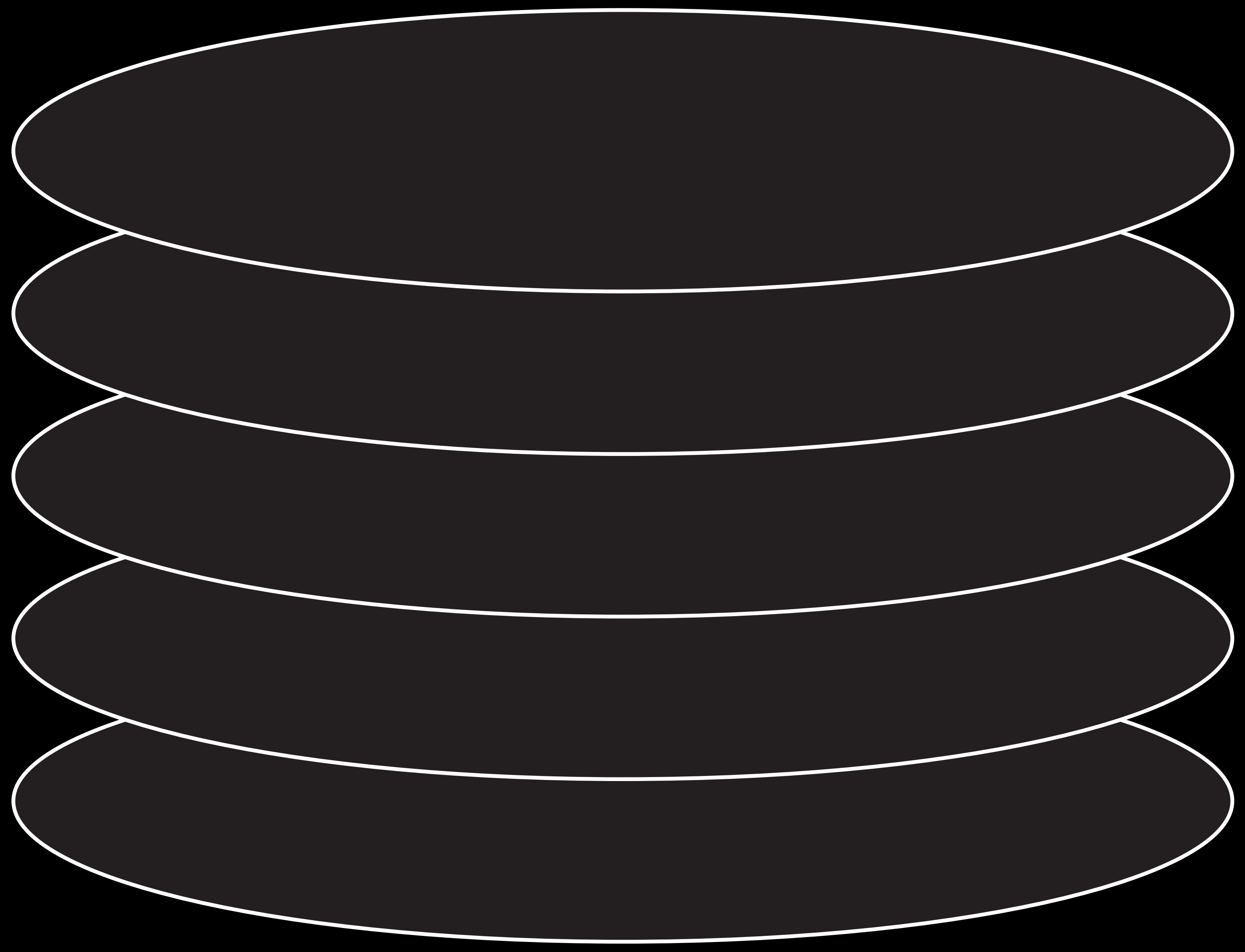
A Ξ-*marked point process* is a probability measure on $\Xi^{\mathbb{M}}$.

If μ is a point process on G, then its *Bernoulli extension* is a $[0,1]$-marked point process denoted $[0,1]^{\mu}$ which arises as the Bernoulli extension of the *Palm equivalence relation* $(\mathbb{M}_0, \mathscr{R}\,|_{\mathbb{M}_0}, \mu_0)$

Informally: sample from μ, then at each point put an IID $\texttt{Unif}[0,1]$ number.

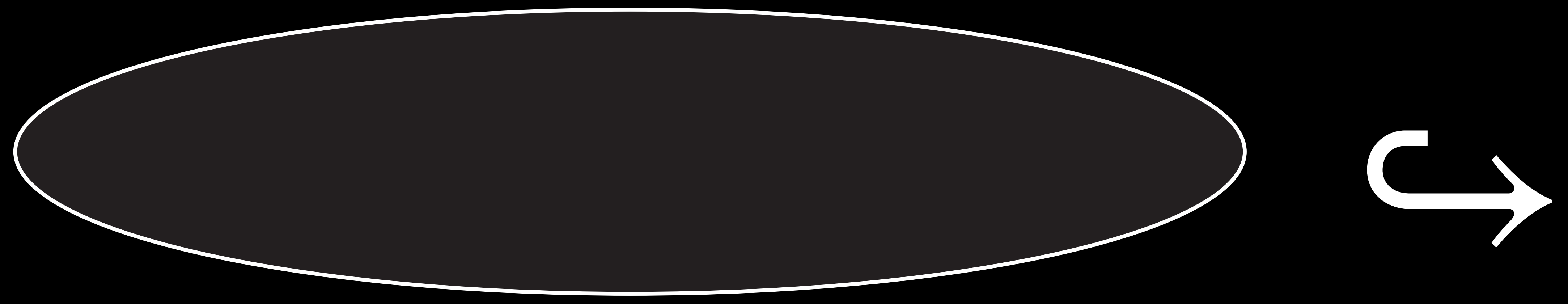
$G \times \mathbb{Z}$ has fixed price one

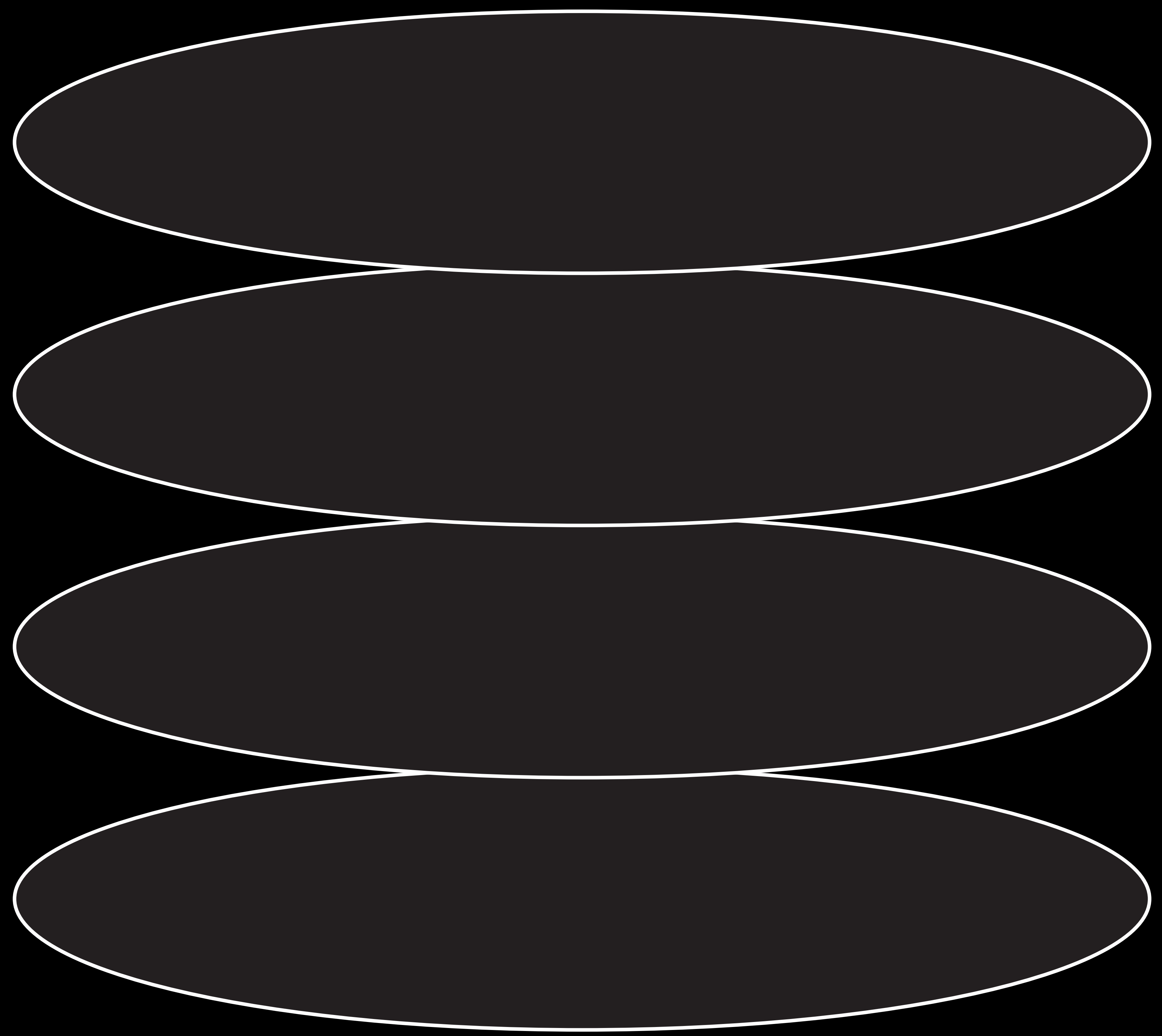
Visualise $G \times \mathbb{Z}$ as an infinite stack of pancakes (or palacsinták, or crêpes, as you prefer)



At least *some* processes have cost one…

Vertical processes

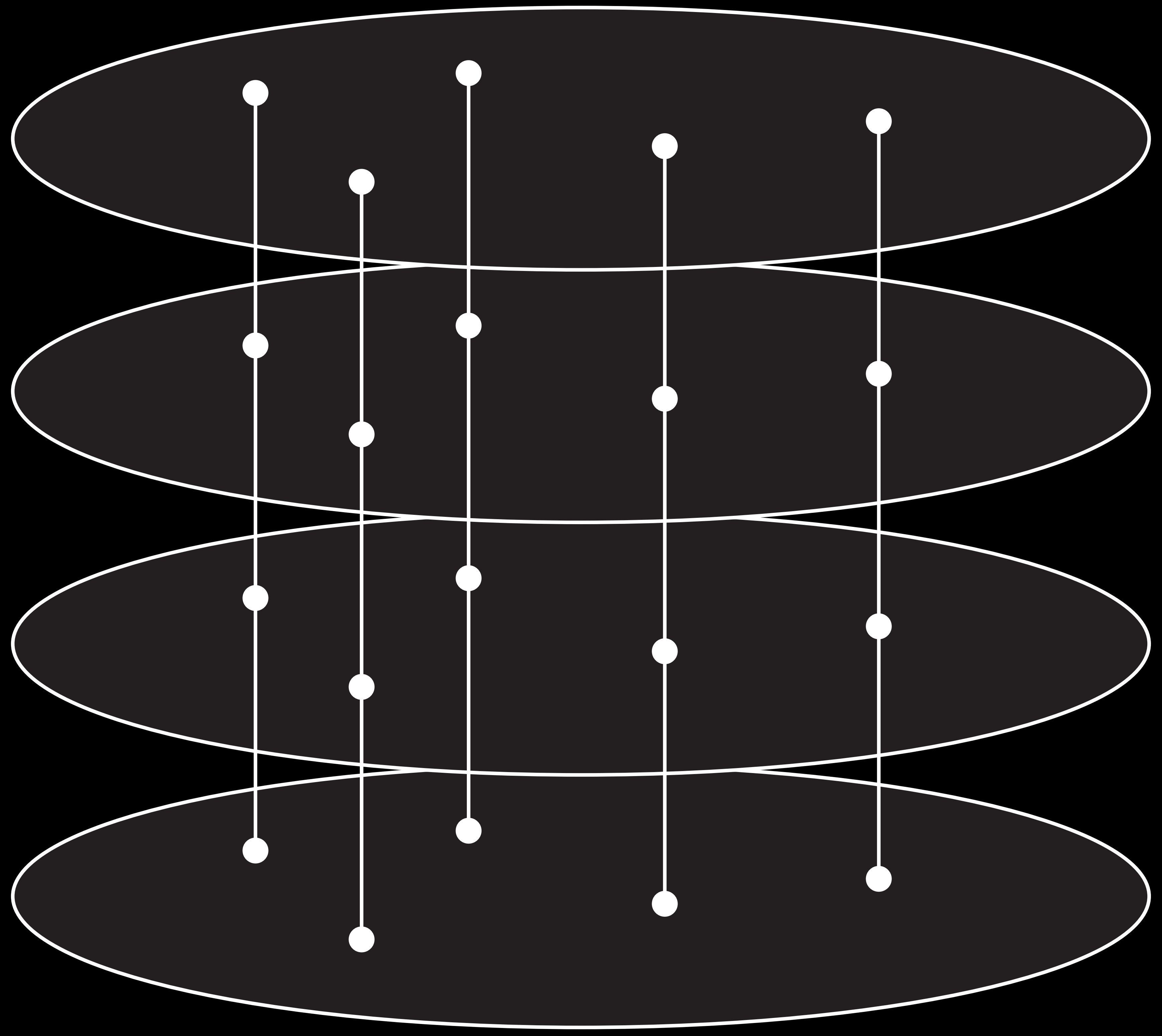




$$\begin{cases} \Delta : \mathbb{M}(G) \to \mathbb{M}(G \times \mathbb{Z}) \\ \Delta(\Pi) = \Pi \times \mathbb{Z} \end{cases}$$

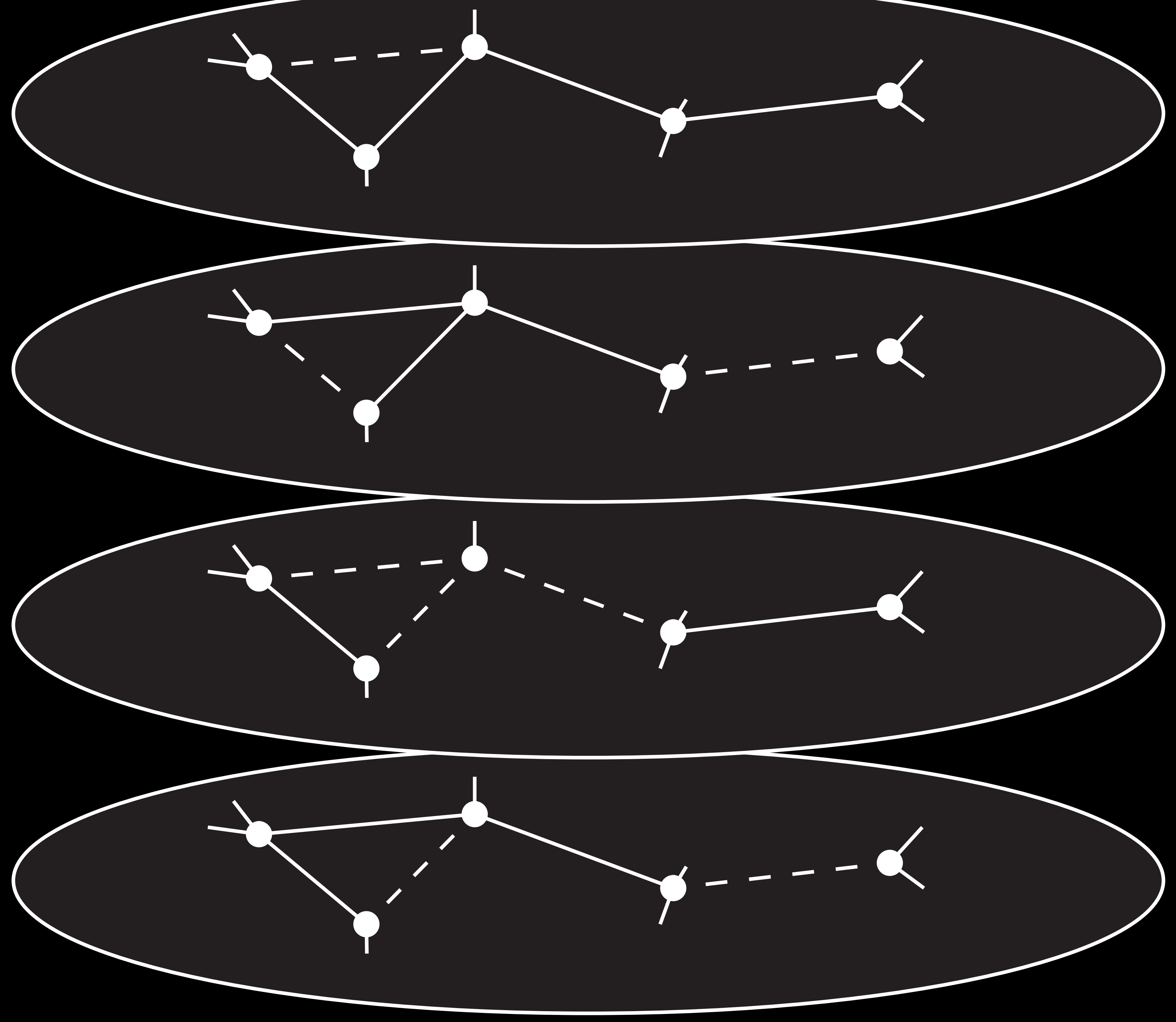
If μ is *any* process on G, then $[0,1]^{\Delta_* \mu}$ has cost one

The cost of Bernoulli extensions of vertical processes



Vertical edges

$\cup$



ε-Percolate any *horizontal* graphing

Fixed price proof outline

- If μ is an invariant point process on $G \times \mathbb{Z}$, then its cost is equal to the cost of its Bernoulli extension $[0,1]^{\mu}$

- *Any* Bernoulli extension $[0,1]^{\mu}$ *weakly factors* onto some vertical process ν *(and onto its Bernoulli extension* $[0,1]^{\nu}$*)*

- Cost is *monotone* for (certain) "weak factors"

Thus $\mathrm{cost}(\mu) = \mathrm{cost}([0,1]^{\mu}) \leq \mathrm{cost}([0,1]^{\nu}) = 1$.

Weak factoring

A point process μ *weakly factors* onto a process ν if there is a sequence of factor maps $\Phi^n : \mathbb{M} \to \mathbb{M}$ such that $\Phi^n_*(\mu)$ *weakly converges* to ν

Inspired by *weak containment* (can probably make this formal)

A sequence of point processes μ^n *weakly converges* to μ if

$$\lim_{n\to\infty} \int_{\mathbb{M}} f(\omega) d\mu_n(\omega) = \int_{\mathbb{M}} f(\omega) d\mu(\omega)$$

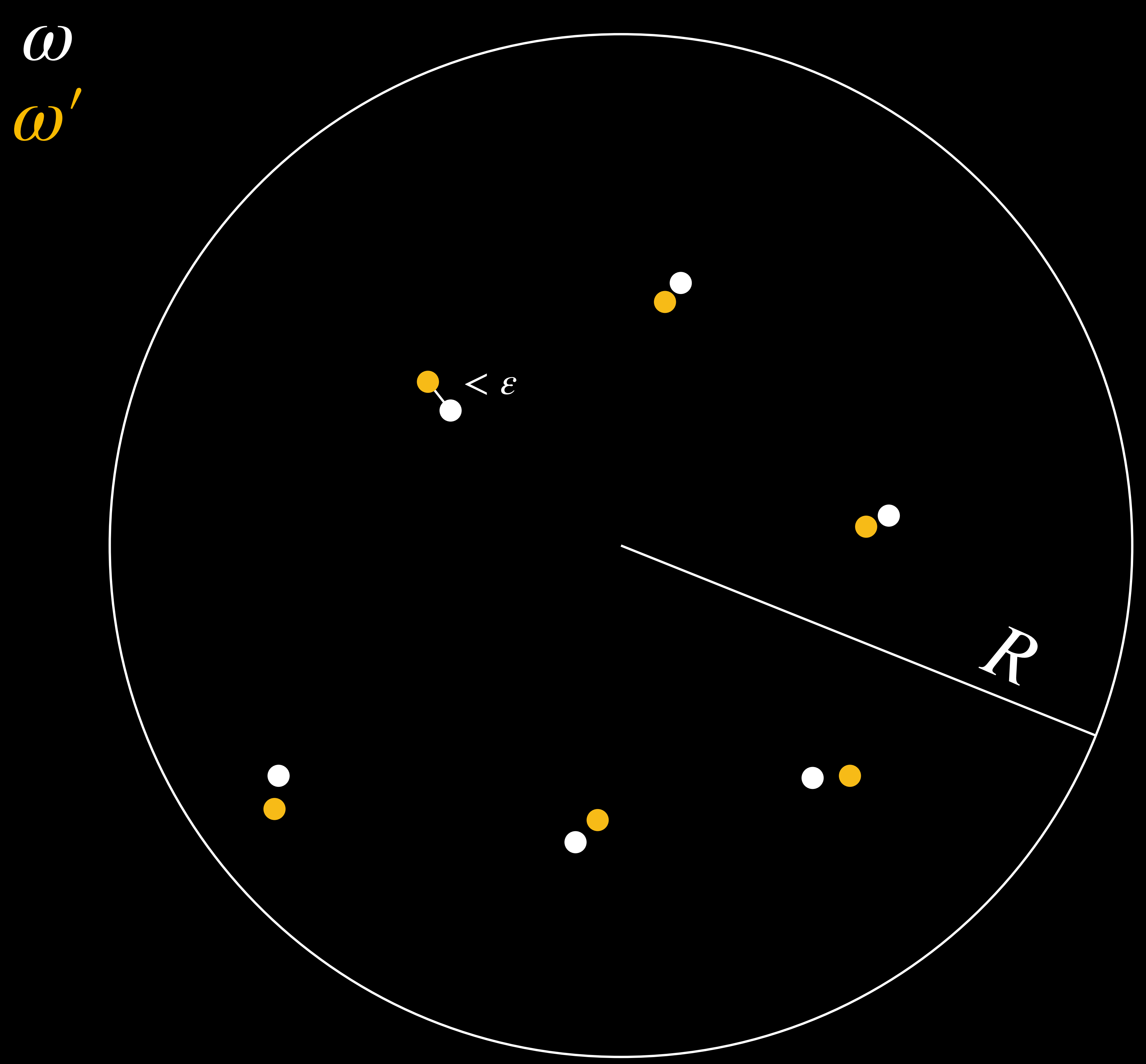
for all *continuous and bounded functions* $f : \mathbb{M} \to \mathbb{R}_{\geq 0}$ with *bounded* support

The topology on $\mathbb{M}$…

…is determined by sequences.

A configuration ω' is an *(R, ε)-wobble* of ω if they are bijectively equivalent in $B(0,R)$ by a bijection $\sigma : \omega' \cap B(0,R) \to \omega \cap B(0,R)$ that moves points by at most ε — that is, $d(x, \sigma(x)) < \varepsilon$

A sequence ω_n *converges* to ω if there exists $R_n \nearrow \infty$ and $\varepsilon \searrow 0$ such that ω_n is an (R_n, ε_n)-wobble of ω



Weak factoring onto a vertical process

Recall that μ is some process on $G \times \mathbb{Z}$.

We wish to weakly factor its Bernoulli extension $[0,1]^\mu$ onto some vertical process.

We take *sparser* and *sparser* subsets of μ and stretch them.

I call this *propagation*.

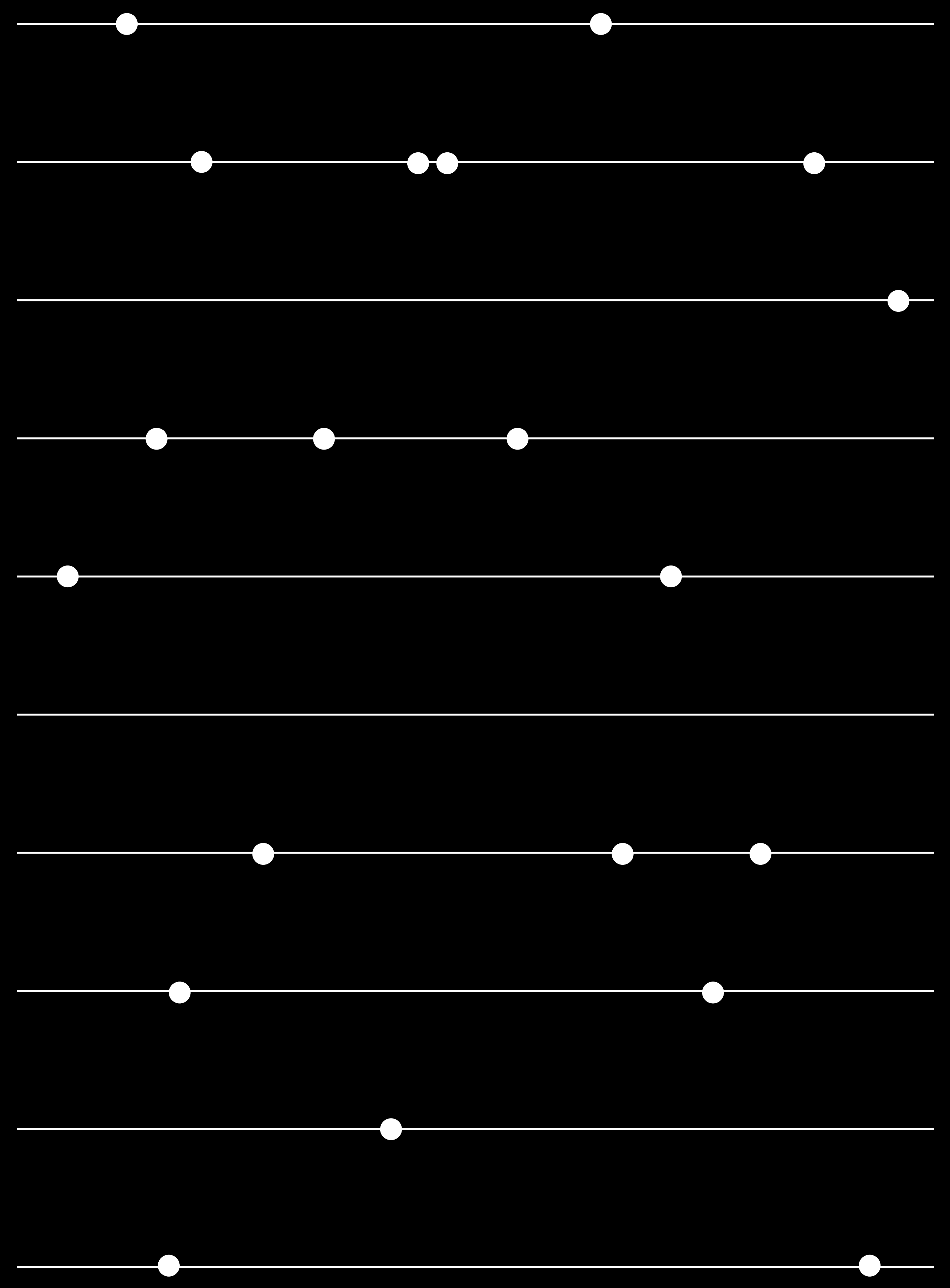
Formally, for $\omega \in [0,1]^{\mathbb{M}(G\times\mathbb{Z})}$, define

$\omega^{1/n} = \{g \in \omega \mid \xi_g < 1/n\}$, and

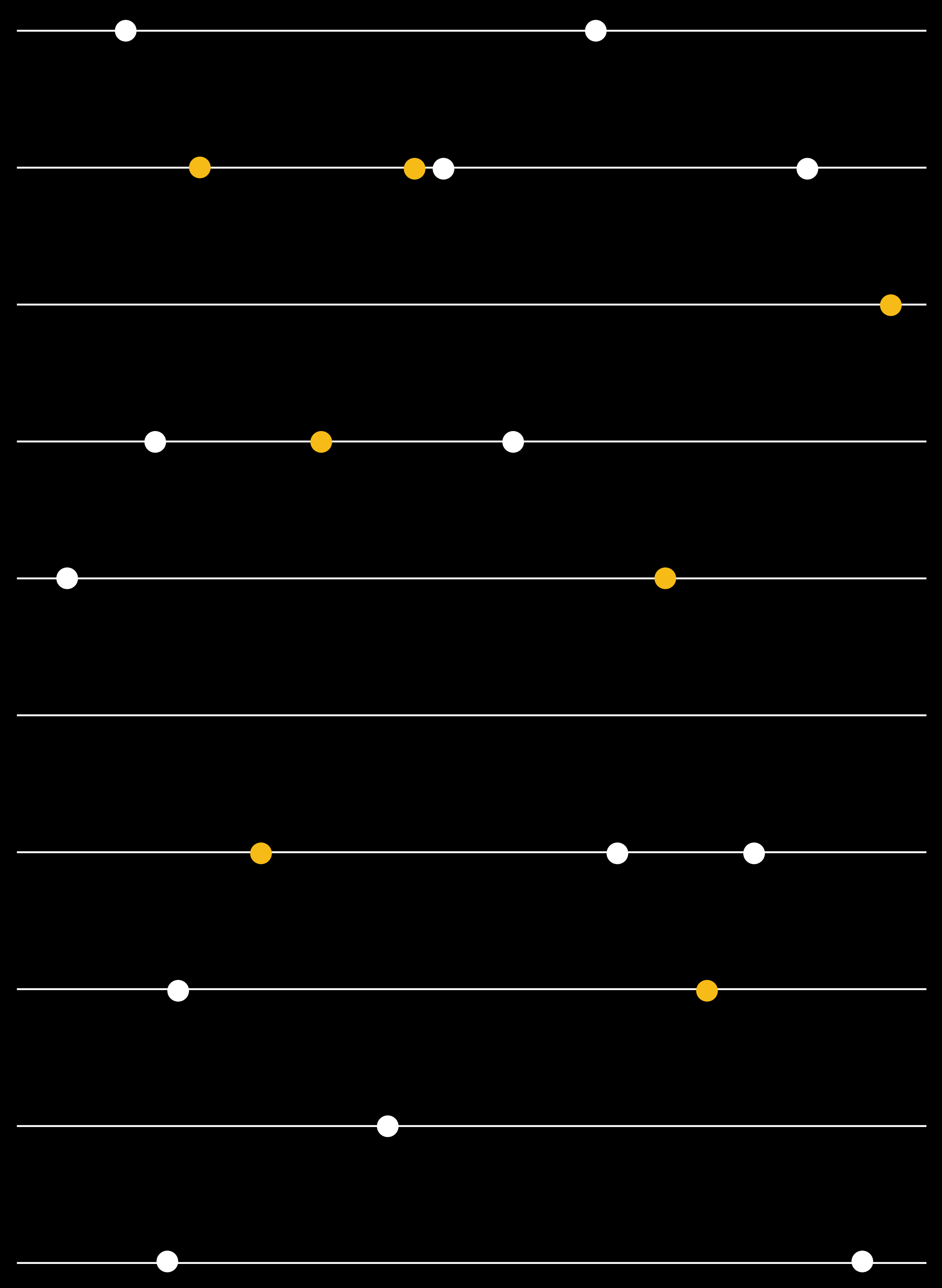
$\omega + 1 = \{(g, l+1) \in G \times \mathbb{Z} \mid (g, l) \in \omega\}$, and

$\Phi^n(\omega) = \omega^{1/n} \cup (\omega^{1/n} + 1) \cup \cdots \cup (\omega^{1/n} + n - 1)$

The *propagation* factor map



The *propagation* factor map



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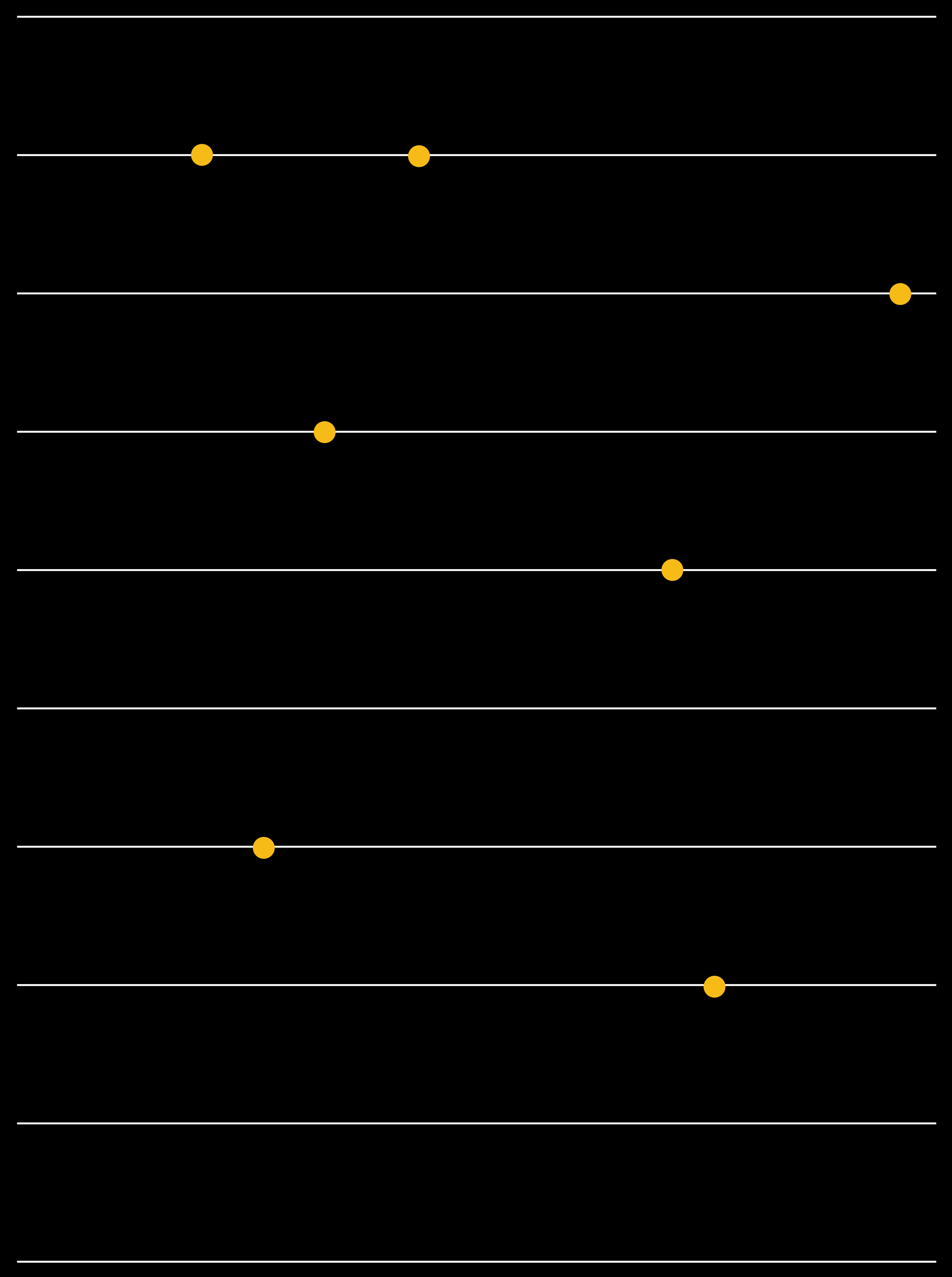

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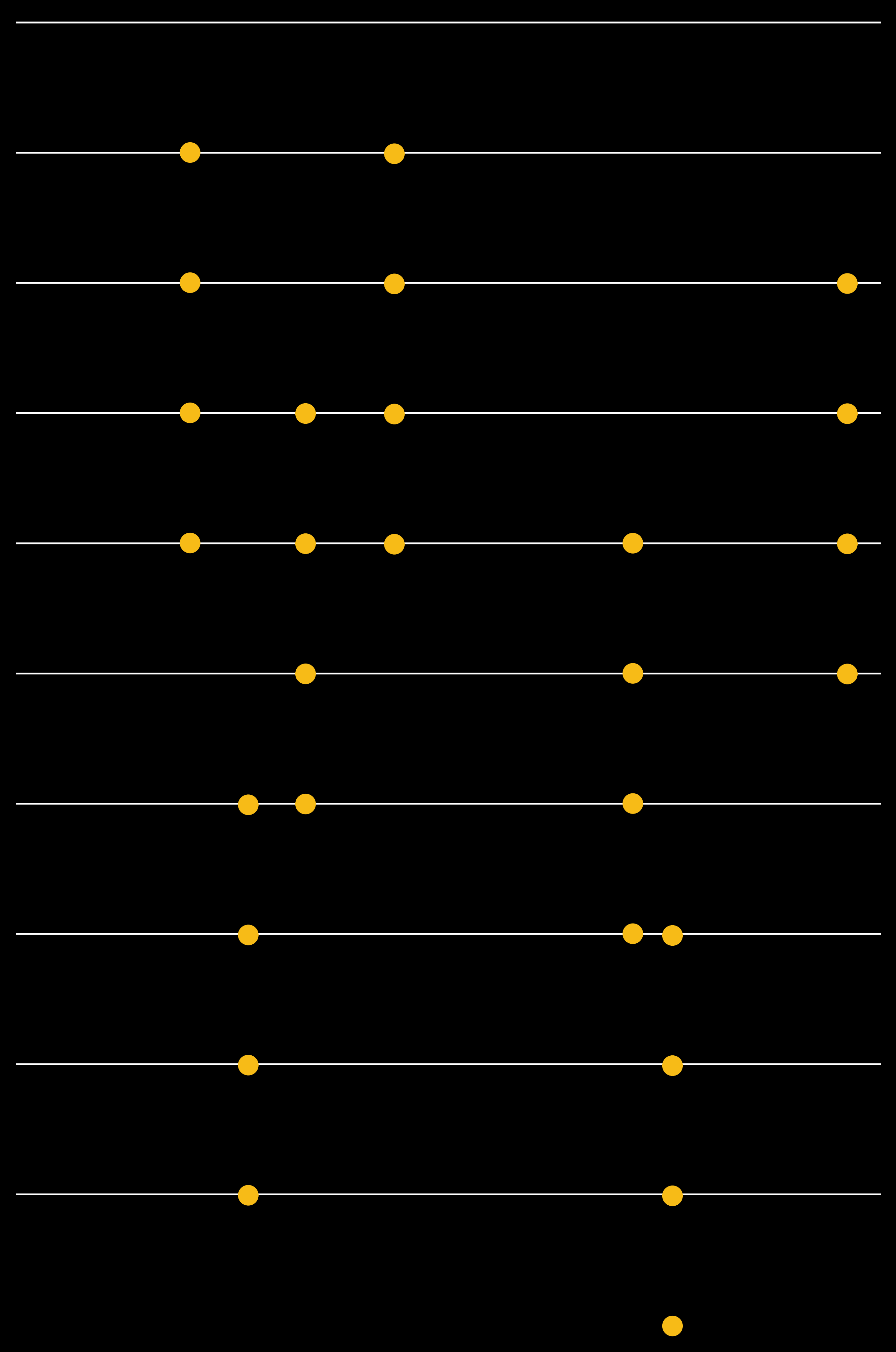

Extensions

- Similar argument covers $G \times \mathbb{R}$
- Modification using Følner sets proves groups containing noncompact amenable normal subgroups have fixed price one
- Can handle $G \times \Lambda$, where Λ is a f.g. group containing an ∞ order element

Question
Do groups of the form $G \times H$ have fixed price one if H contains an ∞ order element generating a discrete subgroup?

Would have interesting ramifications for rank gradient in $SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$.

Replacements…

Replacements…

There is an effort to *replace* all this point process theory by *ultrapowers*.

Fin.

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